

April, 1995

RI-4-95
hep-th/9504080

More Results in $N = 1$ Supersymmetric Gauge Theories

S. Elitzur,¹ A. Forge,² A. Giveon,³ E. Rabinovici⁴

*Racah Institute of Physics, The Hebrew University
Jerusalem, 91904, Israel*

ABSTRACT

We present the exact effective superpotentials in $4d$, $N = 1$ supersymmetric $SU(2)$ gauge theories with N_3 triplets and N_2 doublets of matter superfields. For the theories with a single triplet matter superfield we present the exact gauge couplings for *arbitrary* bare masses and Yukawa couplings.

¹e-mail address: elitzur@vms.huji.ac.il

²e-mail address: forge@vms.huji.ac.il

³e-mail address: giveon@vms.huji.ac.il

⁴e-mail address: eliezer@vms.huji.ac.il

Recently, many new exact results were derived in four dimensional supersymmetric field theories (for a review, see ref. [1]). In this note we report the results ⁵ of applying the methods of refs. [1, 3, 4, 5] to the general case of an infra-red non-trivial $N = 1$ supersymmetric gauge theory with an $SU(2)$ gauge group, N_3 matter supermultiplets in the adjoint representation, Φ_α^{ab} , $\alpha = 1, \dots, N_3$, and $N_2 = 2N_f$ supermultiplets in the fundamental representation, Q_i^a , $i = 1, \dots, 2N_f$. Here a, b are fundamental representation indices, and $\Phi^{ab} = \Phi^{ba}$. To preserve asymptotic freedom or conformal invariance, we need to impose negative or vanishing beta functions; it implies the necessary condition:

$$b_1 = 6 - N_f - 2N_3 \geq 0, \quad (1)$$

where $-b_1$ is the one-loop coefficient of the gauge coupling beta-function. We consider these models in the presence of Yukawa couplings, λ , and masses m . The effective potential is obtained by what is called the “integrating in” method [3, 4]. Under certain conditions, one may, unconventionally, derive the effective superpotential for modes which are of finite mass, given the effective action in which the modes have been considered to have infinite mass. We apply the integrating in technique when it is valid [4]; the various consistency checks to which the result is subjected strengthen the reliability of the method. We integrate in matter in the adjoint representation given the exact effective action for $N_3 = 0$ and $2N_f$ doublets⁶. We obtain the superpotential⁷

$$\begin{aligned} W_{N_f, N_3}(M, X, Z) = & -(4 - b_1) \left\{ \Lambda^{-b_1} \text{Pf} X \left[\det_{N_3}(\Gamma_{\alpha\beta}) \right]^2 \right\}^{1/(4-b_1)} \\ & + \text{Tr}_{N_3} \tilde{m} M + \frac{1}{2} \text{Tr}_{N_2} m X + \frac{1}{\sqrt{2}} \text{Tr}_{N_2} \lambda^\alpha Z_\alpha, \end{aligned} \quad (2)$$

⁵ A more detailed derivation of the results will be furnished in [2].

⁶ The $N_3 = 0$, $2N_f$ superpotential can also be derived by integrating in doublets to the pure $SU(2)$, $N = 1$ supersymmetric Yang-Mills theory [3, 4].

⁷ When $b_1 = 4$ one also obtains constraints; they will be discussed soon. When $b_1 = 0$, “ Λ^{-b_1} ” in (2) should be replaced by a function of $\tau_0 = \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}$ (the non-Abelian gauge coupling) and $\det \lambda$.

where

$$\Gamma_{\alpha\beta}(M, X, Z) = M_{\alpha\beta} + \text{Tr}_{N_2}(Z_\alpha X^{-1} Z_\beta X^{-1}). \quad (3)$$

Here Λ is the dynamically generated scale, while $\tilde{m}_{\alpha\beta}$, m_{ij} and λ_{ij}^α are the bare masses and Yukawa couplings, respectively ($\tilde{m}_{\alpha\beta} = \tilde{m}_{\beta\alpha}$, $m_{ij} = -m_{ji}$, $\lambda_{ij}^\alpha = \lambda_{ji}^\alpha$). The gauge singlets, X , M , Z , are given in terms of the $N = 1$ superfield doublets, Q^a , and the triplets Φ^{ab} , as follows:

$$\begin{aligned} X_{ij} &= \epsilon_{ab} Q_i^a Q_j^b, & a, b = 1, 2, & \quad i, j = 1, \dots, N_2 = 2N_f, \\ M_{\alpha\beta} &= \epsilon_{aa'} \epsilon_{bb'} \Phi_\alpha^{ab} \Phi_\beta^{a'b'}, & \alpha, \beta = 1, \dots, N_3, \\ Z_{ij}^\alpha &= \epsilon_{aa'} \epsilon_{bb'} Q_i^a \Phi_\alpha^{a'b'} Q_j^b. \end{aligned} \quad (4)$$

From eq. (4) it is clear that the determinant in W_{N_f, N_3} vanishes classically, namely: $\Gamma_{\alpha\beta}(M, X, Z) = 0$ is a classical constraint. Quantum mechanically, the constraint is removed; by taking the $\Lambda \rightarrow 0$ limit in eq. (2), one recovers the classical constraint $\det_{N_3}(\Gamma_{\alpha\beta}) = 0$ (if $b_1 < 4$).

The first part of W in (2) is the main result of this paper; it is the exact non-perturbative superpotential. The superpotential is expressed in terms of particular combinations of the gauge singlets M, X, Z . Among other things, it contains the information necessary to derive various subsequent results in this paper.

We consider the general case for the masses of the superfields Q^a . They become massless if $m = \lambda = 0$. In case the doublets are massive, one can obtain the low-energy effective action for the superfields M by integrating out the singlets X and Z from W .

Models without triplets ($N_3 = 0$) were studied in [6, 7]. The superpotential is

$$W_{N_f, 0}(X) = (2 - N_f) \Lambda^{\frac{6-N_f}{2-N_f}} (\text{Pf} X)^{\frac{1}{N_f-2}} + \frac{1}{2} \text{Tr}_{N_2} m X. \quad (5)$$

For $N_f = 1$, the massless superpotential reads: $W = \Lambda^5/X$. For $N_f = 2$ ($b_1 = 4$ in eq. (1)), $W = 0$, and by the integrating in procedure we also get the constraint: $\text{Pf} X = \Lambda^4$. For $N_f > 2$, W is proportional to some positive power of the classical constraint: $\text{Pf} X = 0$.

Models without doublets ($N_f = 0$) were studied in [8, 5, 9]. In these cases

$$W_{0,N_3}(M) = 2(1 - N_3)\Lambda^{\frac{N_3-3}{N_3-1}}(\det M)^{\frac{1}{N_3-1}} + \text{Tr}_{N_3}\tilde{m}M. \quad (6)$$

The massless $N_3 = 1$ case is a pure $SU(2)$, $N = 2$ supersymmetric Yang-Mills theory. This model was considered in detail in ref. [8]. In this case, $W = 0$ (compatible with eq. (2)). As in the other $b_1 = 4$ case, discussed above, by the integrating in procedure one also gets a constraint in this case: $M = \pm\Lambda^2$. This result can be understood as the starting point of the integrating in procedure is a pure $N = 1$ supersymmetric Yang-Mills theory. Therefore, it leads us to the points at the edge of confinement in the moduli space. These are the two singular points in the M moduli space of the theory; they are due to massless monopoles or dyons. Such excitations are not constructed out of the elementary degrees of freedom and, therefore, there is no trace for them in W . (This situation is different if $N_f \neq 0$; in this case, monopoles are different manifestations of the elementary degrees of freedom.)

The $N_f = 0$, $N_3 = 2$ case is discussed in refs. [5, 9]. In this case, the superpotential in eq. (2) is the one presented in [5, 9] on the confining and the oblique confinement branches. As in the $N_3 = 1$ case, this is because the starting point of the integrating in procedure is a pure $N = 1$ supersymmetric Yang-Mills theory and, therefore, it leads us to the confining branches in the moduli space.

For $N_3 = 3$ there is an additional Yukawa coupling that we did not consider in (4): the one which couples the three (antisymmetric) triplets. Therefore, we should also integrate in the additional gauge singlet $\Phi\Phi\Phi \equiv \det\Phi$. The superpotential in eq. (6) remains valid also in the presence of $W_{\text{tree}} = \lambda\det\Phi$ because $\det\Phi = (\det M)^{1/2}$; the Yukawa coupling, λ , replaces “ Λ^0 ” in eq. (6). This result coincides with the one derived in [9]. In the massless case, this theory flows to an $N = 4$ supersymmetric Yang-Mills fixed point.

In the rest of this note we consider the models with a single adjoint matter: $N_3 = 1$, and with fundamental matter: $N_f \neq 0$. In this case M is a complex modulus. All these models have a coulomb phase and thus an

effective Abelian gauge field coupling, $\tau(M, m, \lambda) = \theta/\pi + 8\pi i/g^2$, can be defined for them. The complexified gauge coupling depends on the modulus superfield, as well as the bare masses and Yukawa couplings. The quantum theory is invariant under the $SL(2, Z)$ duality transformations acting on τ [10] and, therefore, it is convenient to define $\tau(M, m, \lambda)$ by the elliptic curve equation:

$$y^2 = x^3 + a(M, m, \lambda)x^2 + b(M, m, \lambda)x + c(M, m, \lambda). \quad (7)$$

We can use the superpotential $W_{N_f,1}$ in (2) in order to find τ . This is done as follows. As was mentioned before, for $N_f > 0$, all the degrees of freedom that may become massless somewhere in the M moduli space are already present in the superpotential. Therefore, the solutions to the equations of motion, derived from W by variations with respect to X and Z , must coincide with the singularities of the elliptic curve (7). This is because for values of M which extremize W , some charged massive modes become massless, and thus give rise to these singularities. In this way we can derive the coefficients a, b, c from W . For $N_f = 1$ this was already done in ref. [5]; one finds

$$a = -M, \quad b = \frac{\Lambda^3}{4}m, \quad c = -\frac{\alpha}{16}, \quad (8)$$

where

$$\alpha \equiv \frac{\Lambda^{2b_1}}{2^{2N_f}} \det \lambda = \frac{\Lambda^6}{4} \det \lambda. \quad (9)$$

For $N_f = 2$ one finds

$$\begin{aligned} a &= -M, & b &= -\frac{\alpha}{4} + \frac{\Lambda^2}{4} \text{Pfm}, \\ c &= \frac{\alpha}{8} (2M + \text{Tr}(\mu^2)), \end{aligned} \quad (10)$$

where

$$\alpha \equiv \frac{\Lambda^{2b_1}}{2^{2N_f}} \det \lambda = \frac{\Lambda^4}{16} \det \lambda, \quad \mu = \lambda^{-1}m. \quad (11)$$

Equation (10) generalizes the result of ref. [11] to *arbitrary* bare masses and Yukawa couplings. Indeed, in the $N = 2$ supersymmetric case (namely, when

$\lambda = \text{diag}(\lambda_1, \lambda_2)$, where λ_1, λ_2 are 2×2 matrices with $\det \lambda_1 = \det \lambda_2 = 1$, and $m = \text{diag}(m_1 \epsilon, m_2 \epsilon)$, where ϵ is the standard 2×2 constant antisymmetric matrix), the result (10) coincides with the one obtained in ref. [11]. All the symmetries and quantum numbers of the various parameters, as used in [8, 11], are already embodied in the superpotential W of eq. (2).

We have also used the same procedure to derive the elliptic curves in the $N_f = 3, 4$; $N_3 = 1$ cases, for *arbitrary* Yukawa couplings and masses. For $N_f = 3$ one finds

$$\begin{aligned} a &= -M - \alpha, \\ b &= 2\alpha M + \frac{\alpha}{2} \text{Tr}(\mu^2) + \frac{\Lambda}{4} \text{Pf} m, \\ c &= \frac{\alpha}{8} \left(-8M^2 - 4M \text{Tr}(\mu^2) - [\text{Tr}(\mu^2)]^2 + 2\text{Tr}(\mu^4) \right), \end{aligned} \quad (12)$$

where

$$\alpha \equiv \frac{\Lambda^{2b_1}}{2^{2N_f}} \det \lambda = \frac{\Lambda^2}{64} \det \lambda, \quad \mu = \lambda^{-1} m. \quad (13)$$

For $N_f = 4$ one finds

$$\begin{aligned} a &= \frac{1}{\beta^2} \left\{ 2 \frac{\alpha+1}{\alpha-1} M + \frac{8}{\beta^2} \frac{\alpha}{(\alpha-1)^2} \text{Tr}(\mu^2) \right\}, \\ b &= \frac{1}{\beta^4} \left\{ -16 \frac{\alpha}{(\alpha-1)^2} M^2 + \frac{32}{\beta^2} \frac{\alpha(\alpha+1)}{(\alpha-1)^3} M \text{Tr}(\mu^2) \right. \\ &\quad \left. - \frac{8}{\beta^4} \frac{\alpha}{(\alpha-1)^2} [(\text{Tr}(\mu^2))^2 - 2\text{Tr}(\mu^4)] + \frac{4}{\beta^4} \frac{(\alpha+1)\Lambda^{b_1}}{(\alpha-1)^2} \text{Pf} m \right\}, \\ c &= \frac{1}{\beta^6} \left\{ -32 \frac{\alpha(\alpha+1)}{(\alpha-1)^3} M^3 + \frac{32}{\beta^2} \frac{\alpha(\alpha+1)^2}{(\alpha-1)^4} M^2 \text{Tr}(\mu^2) \right. \\ &\quad \left. + M \left[-\frac{16}{\beta^4} \frac{\alpha(\alpha+1)}{(\alpha-1)^3} ((\text{Tr}(\mu^2))^2 - 2\text{Tr}(\mu^4)) + \frac{32}{\beta^4} \frac{\alpha\Lambda^{b_1}}{(\alpha-1)^3} \text{Pf} m \right] \right. \\ &\quad \left. - \frac{32}{\beta^6} \frac{\alpha}{(\alpha-1)^2} [\text{Tr}(\mu^2) \text{Tr}(\mu^4) - \frac{1}{6} (\text{Tr}(\mu^2))^3 - \frac{4}{3} \text{Tr}(\mu^6)] \right\}. \end{aligned} \quad (14)$$

Here α and β are functions of τ_0 , the non-Abelian gauge coupling constant;

comparison with ref. [11] gives

$$\alpha(\tau_0) \equiv \frac{\Lambda^{2b_1}}{2^{2N_f}} \det \lambda = \left(\frac{\theta_2^2 - \theta_3^2}{\theta_2^2 + \theta_3^2} \right)^2, \quad \beta(\tau_0) = \frac{\sqrt{2}}{\theta_2 \theta_3}, \quad \mu = \lambda^{-1} m, \quad (15)$$

where

$$\theta_2(\tau_0) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i \tau_0 n^2}, \quad \theta_3(\tau_0) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau_0 n^2}, \quad \tau_0 = \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}. \quad (16)$$

($16\alpha^{1/2}(\det \lambda)^{-1/2}$ replaces “ Λ^{b_1} ” in eq. (2); $\alpha(\tau_0)$ is dimensionless, and has zero $U(1)_R \times U(1)_Q \times U(1)_\Phi$ quantum numbers.)⁸

In eq. (14) M is rescaled with respect to M in the superpotential, $M \rightarrow \beta^2 M$, using the scale invariance of the $N = 2$ theory with four flavors⁹.

The S -duality symmetry is valid in the $N_3 = 1, N_f = 4$ theories for *arbitrary* λ, m , similar to the $SL(2, \mathbb{Z})$ invariance in the presence of masses discussed in ref. [11]. The $SL(2, \mathbb{Z})$ transformations map τ_0 to $(a\tau_0 + b)(c\tau_0 + d)^{-1}$, $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$. Combined with triality (which acts on μ), it leaves the elliptic curve invariant.

Equations (12, 14) generalize the results obtained in [11] for the $N = 2$ supersymmetric case (namely, for an appropriate subspace of m and λ). Needless to say that by taking the mass m_{N_2, N_2-1} to infinity, we can generate the effective superpotential with $N_f - 1$ flavors from the solution with N_f flavors (by integrating out), as well as the corresponding elliptic curve.

We shall end with a few remarks:

- The derivation of the elliptic curves from the superpotentials, in all $N_3 = 1$ cases ($N_f = 1, 2, 3, 4$ in eqs. (8), (10), (12), (14)), suggests that

⁸ $(\det \lambda)^{-1/2}$ has the correct quantum numbers needed for the matching condition, $\alpha^{1/2}(\det \lambda)^{-1/2} \tilde{m} = \Lambda_{N_f=4, N_3=0}$, which we used in the integrating in procedure. To compare eq. (14) with ref. [11] we need to take $m = \text{diag}(m_1 \epsilon, m_2 \epsilon, m_3 \epsilon, m_4 \epsilon)$ and $\lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where λ_I , $I = 1, 2, 3, 4$, are 2×2 matrices with $\det(\lambda_I) = 1$. In this case, $\text{Tr}(\mu^2) = -2 \sum_{I=1}^4 m_I^2$, $(\text{Tr}(\mu^2))^2 - 2\text{Tr}(\mu^4) = 8 \sum_{I < J} m_I^2 m_J^2$, $\text{Tr}(\mu^2)\text{Tr}(\mu^4) - \frac{1}{6}(\text{Tr}(\mu^2))^3 - \frac{4}{3}\text{Tr}(\mu^6) = 8 \sum_{I < J < K} m_I^2 m_J^2 m_K^2$.

⁹ Note that W in eq. (2) scales appropriately, $W \rightarrow \beta^3 W$, under the scale transformation: $\Phi \rightarrow \beta \Phi, Q \rightarrow \beta Q, m \rightarrow \beta m, \Lambda \rightarrow \beta \Lambda, \lambda \rightarrow \lambda$.

the variable x in eq. (7) could be identified with Γ in eq. (3) (up to a shift by M).

- The techniques used, and the patterns uncovered in this note can be applied also to the $N_3 > 1$ cases (with mass given to part of the M fields), and to other gauge groups. We shall report on that in [2].
- The $SU(2)$, $N_3 = 1$, N_f models fall into a lacuna in the analysis in ref. [12] of the dual models to $SU(N_c)$ systems with matter in the adjoint and fundamental representations. The results obtained here might shed some light on this gap ¹⁰.
- Finally, to complete the survey of models obeying eq. (1), let us note that one can also have an infra-red non-trivial theory with a single matter superfield in the $I = 3/2$ representation ($N_4 = 1$ in our notation). The $N_4 = 1$, $N_f = 0$ theory was shown to have $W = 0$ [13]. Adding $N_f = 1$ matter results with $b_1 = 0$ in eq. (1). The two-loop beta function renders the theory infra-red free. As no Yukawa coupling is possible, this model is indeed infra-red free.

Acknowledgements

We thank S. Forste and N. Seiberg for discussions. The work of SE is supported in part by the BRF - the Basic Research Foundation. The work of AG is supported in part by BSF - American-Israel Bi-National Science Foundation, by the BRF, and by an Alon fellowship. The work of ER is supported in part by BSF and by the BRF.

¹⁰We thank N. Seiberg for a discussion on this point.

References

- [1] N. Seiberg, hep-th/9408013.
- [2] S. Elitzur, A. Forge, A. Givon and E. Rabinovici, in preparation.
- [3] K. Intriligator, R.G. Leigh and N. Seiberg, Phys. Rev. **D50** (1994) 1092.
- [4] K. Intriligator, Phys. Lett. **B336** (1994) 409.
- [5] K. Intriligator and N. Seiberg, Nucl. Phys. **B431** (1994) 551.
- [6] G. Veneziano and S. Yankielowicz, Phys. Lett. **B113** (1982) 321; T.R. Taylor, G. Veneziano and S. Yankielowicz, Nucl. Phys. **B218** (1983) 493; A.C. Davis, M. Dine and N. Seiberg, Phys. Lett. **B125** (1983) 487; I. Affleck, M. Dine and N. Seiberg, Nucl. Phys. **B241** (1984) 493, Nucl. Phys. **B256** (1985) 557; D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, Phys. Rep. **162** (1988) 169, and references therein.
- [7] N. Seiberg, Phys. Rev. **D49** (1994) 6857.
- [8] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19.
- [9] K. Intriligator and N. Seiberg, hep-th/9503179.
- [10] J. Cardy and E. Rabinovici, Nucl. Phys. **B205** (1982) 1; J. Cardy, Nucl. Phys. **B205** (1982) 17.
- [11] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994) 484.
- [12] D. Kutasov, hep-th/9503086.
- [13] K. Intriligator, N. Seiberg and S.H. Shenker, Phys. Lett. **B342** (1995) 152.